ADEQUATE MODELING CRYPTOCURRENCIES PRICES

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Outline

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Historical background

Recently volatility of cryptocurrency market has increased essentially. Investors have a need for adequate modeling of the key cryptocurrencies prices. Lévy processes are more flexible than Gaussian models. These models provide a better fit to empirical asset price distributions and admit jumps in asset prices.

Methods for calibration in financial markets

- Moment matching: *historical measure*
- Fitting parameters to option prices: *equivalent martingale measure*

The main goal

To suggest a new approach for calibration of Lévy models to cryptocurrencies prices.

The key ideas

Activity of jumps analysis

On the basis of time series of log-returns of historical cryptocurrency Bitcoin prices, the jump activity index is estimated. The singularity of Lévy measure is described. Choose an adequate Lévy model.

Crossing barrier events: frequency estimation

- Set upward and downwards barriers for the current log-price: $\pm 0.01, \pm 0.02, \ldots, \pm 0.3$.
- For each time frame (1 day, 5 days, etc) in a given period: compute the maximal and the minimal log-returns and check which barriers are crossed;
- For each barrier: estimate the frequency of crossing

Fitting parameters

Fit parameters to artificial first touch digital option prices

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Lévy processes: a short reminder

General definitions

A Lévy process is a stochastically continuous process with stationary independent increments (for general definitions, see e.g. Sato (1999)). A Lévy process can be completely specified by its characteristic exponent, ψ, definable from the equality $E[e^{i\xi X(t)}] = e^{-t\psi(\xi)}$.

The characteristic exponent of Lévy process

The characteristic exponent is given by the Lévy-Khintchine formula:

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \int_{-\infty}^{+\infty}(1 - e^{i\xi y} + i\xi y \mathbf{1}_{|y|\leq 1})F(dy),$$

where σ^2 is the variance of the Gaussian component, and the Lévy measure $F(dy)$ satisfies $\int_{\mathbf{R}\setminus\{0\}} \min\{1, y^2\}F(dy) < +\infty$.

Examples of Lévy processes

Tempered stable Lévy processes (TSL)

$$\psi(\xi) = -i\mu\xi + c_+\Gamma(-\nu_+)[\lambda_+^{\nu_+} - (\lambda_+ + i\xi)^{\nu_+}] + c_-\Gamma(-\nu_-)[(-\lambda_-)^{\nu_-} - (-\lambda_- - i\xi)^{\nu_-}],$$

where $\nu_+, \nu_- \in (0, 2), \nu_+, \nu_- \neq 1$, $c_+, c_- > 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$. If $c_- = c_+ = c$ and $\nu_- = \nu_+ = \nu$, then we obtain a KoBoL (CGMY) model.
In the CGMY parametrization $C = c$, $Y = \nu$, $G = \lambda_+$, $M = -\lambda_-$.

Kou model

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \frac{ic_+\xi}{\lambda_+ + i\xi} + \frac{ic_-\xi}{\lambda_- + i\xi},$$

where $c_+, c_- \geqslant 0$, $\mu \in \mathbf{R}$, and $\lambda_- < -1 < 0 < \lambda_+$.

Wiener-Hopf factorization

Let $q > 0$, X_t be a Lévy process with characteristic exponent $\psi(\xi)$, $T_q \sim \mathrm{Exp}\, q$, $\bar{X}_t = \sup_{0 \leqslant s \leqslant t} X_s$ and $\underline{X}_t = \inf_{0 \leqslant s \leqslant t} X_s$ – supremum and infimum processes.

$$\phi_q^+(\xi) = E[e^{i\xi \bar{X}_{T_q}}], \quad \phi_q^-(\xi) = E[e^{i\xi \underline{X}_{T_q}}], \quad \frac{q}{q + \psi(\xi)} = \phi_q^+(\xi)\phi_q^-(\xi).$$

Introduce the following operators:

$$\mathcal{E}_q g(x) = E^x[\int_0^{+\infty} q e^{-qt} g(X_t) dt] = E^x[g(X_{T_q})].$$

$$\mathcal{E}_q^+ g(x) = E^x[\int_0^{+\infty} q e^{-qt} g(\bar{X}_t) dt] = E^x[g(\bar{X}_{T_q})].$$

$$\mathcal{E}_q^- g(x) = E^x[\int_0^{+\infty} q e^{-qt} g(\underline{X}_t) dt] = E^x[g(\underline{X}_{T_q})].$$

$\mathcal{E}$ and $\mathcal{E}^{\pm}$ as PDO

$$\mathcal{E}_q g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi} q(q+\psi(\xi))^{-1}\hat{g}(\xi)d\xi,$$

$$\mathcal{E}_q^{\pm} g(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ix\xi}\phi_q^{\pm}(\xi)\hat{g}(\xi)d\xi.$$

WHF in an operator form: $\mathcal{E} = \mathcal{E}^{+}\mathcal{E}^{-} = \mathcal{E}^{-}\mathcal{E}^{+}$.

$\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ as convolution operators

Operators $\mathcal{E}_q$ and $\mathcal{E}_q^{\pm}$ admit the following interpretation:

$$\mathcal{E}_q g(x) = \int_{-\infty}^{+\infty} g(x+y)P(y)dy, \quad \mathcal{E}_q^{\pm} g(x) = \int_{-\infty}^{+\infty} g(x+y)P_{\pm}(y)dy,$$

where $P(y)$, $P_{\pm}(y)$ are probability densities with

$$P_{\pm}(y) = 0, \quad \forall \pm y < 0.$$

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Pricing first touch digital options

First touch digital options

Let T, H be the maturity, barrier, and $S_t = He^{X_t}$ be the stock price under a chosen risk-neutral measure. The first touch digital contract with a barrier H pays $1, as a stock price S_t for first time crosses the barrier H. If up to the date T the price does not cross the barrier H, the option becomes worthless.

The Fast Wiener-Hopf factorization method (FWHF-method)

In Kudryavtsev (2016) the fast, accurate and universal numerical method for pricing first touch digital options under Lévy models was developed.

Reference

KUDRYAVTSEV O. Advantages of the Laplace transform approach in pricing first touch digital options in Lévy-driven models. Boletín de la Sociedad Matemática Mexicana, 2016, vol. 22(2), pp. 711–731.

Laplace transform of probability functions

WH method

- Apply Laplace transform to $F_+(x,T) = \mathbf{P}(x + \bar{X}_T > 0)$, $x < 0$:

$$\begin{aligned}
\hat{F}_+(x,q) &= \int_0^{+\infty} e^{-qt} E^x\left[\mathbf{1}_{(0,+\infty)}(\bar{X}_t)\right] dt \\
&= q^{-1} E\left[\mathbf{1}_{(0,+\infty)}(x + \bar{X}_{T_q})\right] \\
&= q^{-1} \mathcal{E}_q^+ \mathbf{1}_{(0,+\infty)}(x)
\end{aligned}$$

- Apply Laplace transform to $F_-(x,T) = \mathbf{P}(x + \underline{X}_T < 0)$, $x > 0$:

$$\begin{aligned}
\hat{F}_-(x,q) &= \int_0^{+\infty} e^{-qt} E^x\left[\mathbf{1}_{(-\infty,0)}(\underline{X}_t)\right] dt \\
&= q^{-1} E\left[\mathbf{1}_{(-\infty,0)}(x + \underline{X}_{T_q})\right] \\
&= q^{-1} \mathcal{E}_q^- \mathbf{1}_{(-\infty,0)}(x)
\end{aligned}$$

Numerical Laplace transform inversion: the Gaver-Stehfest algorithm

An approximate formula for $f(\tau)$ can be written as follows

$$f(\tau) \approx \frac{1}{\tau}\sum_{k=1}^{N} \omega_k \cdot \tilde{f}\left(\frac{\alpha_k}{\tau}\right), \qquad 0 < \tau < \infty,$$

$$\begin{aligned} N &= 2n; \\ \alpha_k &= k\ln(2) \\ \omega_k &:= \frac{(-1)^{n+k}\ln(2)}{n!}\sum_{j=[(k+1)/2)]}^{\min\{k,n\}} j^{n+1} C_n^j C_{2j}^j C_j^{k-j}, \end{aligned}$$

where $[x]$ – integer part x и $C_L^K = \frac{L!}{(L-K)!K!}$ – binomial coefficients.

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Activity of jumps analysis

Activity index

We define activity index

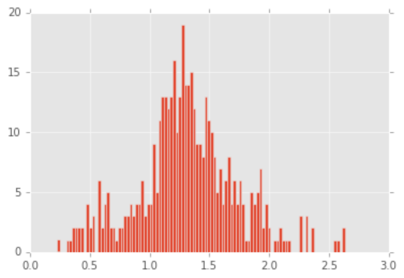
$$\beta_{X,T} := \inf\{p > 0 : \lim_{\Delta_n \to 0} \sum_{i=1}^{T/\Delta_n} |x_{t_i} - x_{t_{i-1}}|^p < \infty\},$$

from the power variations with the order p and the time step $t_i - t_{i-1} = \Delta_n = 1/n$

Reference

TODOROV V., TAUCHEN G. Activity signature functions for high-frequency data analysis. Journal of Econometrics. 2010. vol. 154, № 1, pp. 136-–141.

Activity index distribution - BTC/USD

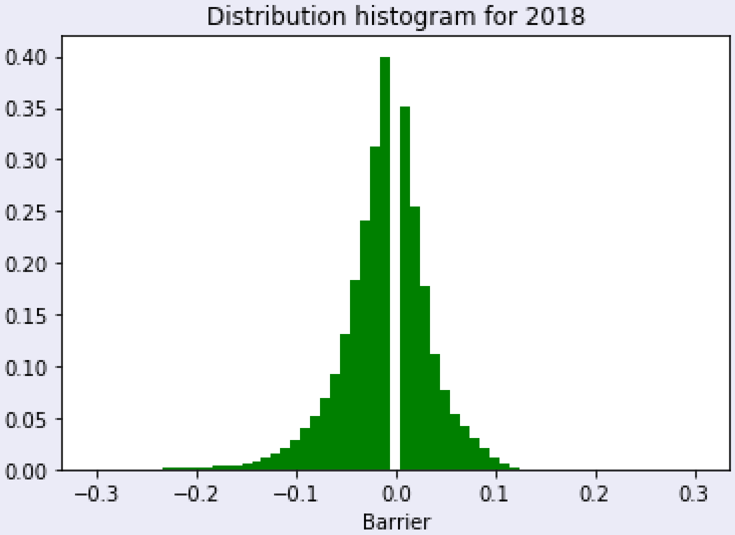


The mean and confidence interval for the activity index BTC/USD :

$$Mean = 1.31094162801, (1.27244, 1.34944).$$

The quantiles analysis of the index showed that the Lévy processes with unbounded variation are most adequate; for example, a well-known CGMY (KoBoL) model, the parameter Y of which can be estimated by the activity index.

Crossing barrier frequency distribution - BTC/USD



CGMY model calibration

The restrictions to the parameters

We fix the following restrictions for the parameters:

$$Y \in (1.27244, 1.34944), M, G \in (2, 100), C \in (0.5, 2)$$

The error

We use the Nelder–Mead method algorithm to minimize the error:

$$\epsilon = \sum_i \ln \frac{p_i}{\hat{p}_i}(p_i - \hat{p}_i)$$

The error ϵ takes into account the impact of absolute and relative errors simultaneously.

The results

$$C = 1.005315,\ G = 17.234117,\ M = 6.439021,\ Y = 1.273$$